quite thick due to a large t_2 generally is not appreciated and is quite important. The use of very thin-skinned models in a vain attempt to reduce conduction errors can lead to structural problems and to errors due to heat conduction to the interior of the model, as noted previously.

The measurement of recovery temperature with thin-skinned models now is considered. The skin temperature at large time approaches a steady state near the recovery temperature. By assuming that this steady T is approximately equal to T_r, one may use T_r to evaluate the conduction term in the steady form of Eq. (2). Thus one obtains

$$T - T_r = (\delta k/h)\nabla^2 T_r + (k/h)\nabla\delta\cdot\nabla T_r$$

This equation can be used to estimate the error in a recovery temperature measurement by using the experimental values of T_r to evaluate the right side. Then the criterion for negligible tangential conduction error in recovery temperature measurements is

$$|(\delta k/h)\nabla^2 T_r + (k/h)\nabla\delta\cdot\nabla T_r| \leqq |\text{allowable } T_r \text{ error}|$$

In Ref. 6 a similar result, Eq. (27), is derived more rigorously but is restricted to axisymmetric bodies of uniform skin thickness. Although most investigators use models made of an insulator to measure T_r, if the foregoing criterion is satisfied, then a thin skinned heat transfer model also may be employed to measure T_r.

Each of the errors discussed here can be quite significant in particular experiments. They all should be checked carefully during the design of experiments, particularly when new features such as low or strongly varying heat transfer coefficients are expected.

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Nonlinear Guidance System for Descent Trajectories

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A SURVEY paper on space rendezvous, with many references, appeared in *Astronautica Acta*.[1] Terminal guidance for space rendezvous has been divided into two classes: that based on orbital mechanics[2] in the time domain and that based on proportional navigation.[3, 4] This paper deals mainly with problems of descent trajectories including soft-landing.[5–8]

Received by ARS November 7, 1962; revision received May 27, 1963. The research presented in this paper was supported by NASA under Research Grant No. NsG 14-59.

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1. General Equations of Motion

A point mass m is attracted to a planet by a central force proportional to the inverse square of the distance. The equations of motion are[9]

$$dk/d\theta = 2a_\theta/u^3 \tag{1}$$

$$\frac{d^2u}{d\theta^2} + \frac{dk/d\theta}{2k}\frac{du}{d\theta} + u - \frac{g_0}{ku_0{}^2} = -\frac{a_r}{ku^2} \tag{2}$$

where

$$u = 1/r = \text{inverse of radius vector} \tag{3}$$

$$k = h^2 = [r^2\dot{\theta}]^2 = \text{square of specific angular momentum} \tag{4}$$

and where a_θ, a_r are the transverse and radial specific forces, respectively, and g_0 is the gravitational acceleration at the surface of the planet.

A further transformation is advantageous by virtue of Eq. (1). Let

$$\frac{du}{d\theta} = \frac{dk}{d\theta}\frac{du}{dk} = \left(\frac{2a_\theta}{u^3}\right)\frac{du}{dk} \tag{5}$$

If Eq. (5) and its derivative with respect to θ are substituted into Eq. (2), one has

$$\frac{d^2u}{dk^2} + \left[\frac{u^3}{2a_\theta}\frac{d}{dk}\left(\frac{2a_\theta}{u^3}\right) + \frac{1}{2k}\right]\frac{du}{dk} + \left(\frac{u^3}{2a_\theta}\right)^2\left[u - \frac{g_0}{ku_0{}^2} + \frac{a_r}{ku^2}\right] = 0 \tag{6}$$

and

$$d\theta/dk = u^3/2a_\theta \tag{7}$$

by inverting Eq. (1).

The dependent variable k in Eq. (1) becomes the independent variable k in Eqs. (6) and (7). Equation (6) is a key equation that does not contain θ explicitly. It is linear in u and can be solved independently of Eq. (7), provided that a_θ/u^3 and a_r/u^2 are expressed in terms of k, such as

$$a_\theta = u^3G_\theta(k) \qquad a_r = u^2G_r(k) \tag{8}$$

where $G_\theta(k)$ and $G_r(k)$ are functions of k only.

2. Descent Trajectory

A vehicle in orbit is traveling in the direction $b' - b$, as shown on Fig. 1. At point b, the vehicle begins its descent phase by firing a retrorocket having a specific force with a radial component a_r and transverse component a_θ. The descent trajectory is defined by its polar coordinates (r, θ) measured from the point of soft-landing O.

The measurement of θ can be made by inertial means, e.g., an inertial guidance package with both gyros and accelerometers to determine the local vertical with respect to site vertical. The quantity $u = 1/r$ can be determined by measuring the altitude of the vehicle above the planet. The controlled specific forces a_θ and a_r will be in terms of the known quantities θ and u.

For the virgin landing of a vehicle near an unexplored planet, it is desirable to allow the astronaut to hover the vehicle near the surface. The conditions of hovering are zero velocities and zero accelerations in both the transverse and radial directions.

A. Transverse specific force

In order to obtain zero transverse velocity, the specific angular momentum h_0 at point O should be zero; thus, $k_0 = h_0{}^2 = 0$. The transverse specific force a_θ should be zero so

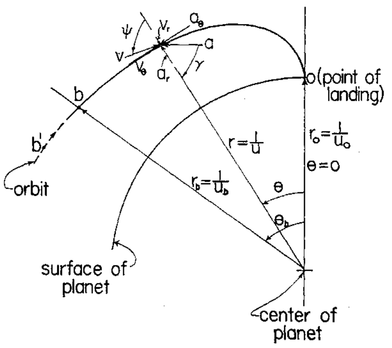

Fig. 1 Descent trajectories.

that the acceleration of the vehicle in the same direction is also zero. Thus, one has

$$a_\theta = (u^3/2\beta)(k/k_b)^n \qquad 0 < n \quad (9)$$

where β is a constant. The power index n should be a positive real number so that a_0 is bounded at $k = 0$.

B. *Specific angular momentum*

If the result of $u^3/2a_\theta$ from Eq. (9) is substituted into Eq. (7) and the integration is performed, one has

$$\theta/\theta_b = k^{1-n}/k_b^{1-n} \quad (10)$$

where k_b is the value of k at $\theta = \theta_b$. Solving for k in terms of θ for $n = \frac{1}{2}$, one obtains

$$k/k_b = \theta^2/\theta_b^2 \quad (11)$$

The specific angular momentum ratio h/h_b may be obtained by combining Eqs. (4) and (11) into a linear relation:

$$h/h_b = \theta/\theta_b \quad (12)$$

C. *Radial specific force*

The differentiation of a_θ/u^3 from Eq. (9) (using $n = \frac{1}{2}$) with respect to k may be obtained and substituted into Eq. (6):

$$\frac{d^2u}{dk^2} + \left(\frac{1}{k}\right)\frac{du}{dk} + \beta^2\left(\frac{k_b}{k}\right)\left(u - \frac{g_0}{ku_0^2} + \frac{a_r}{ku^2}\right) = 0 \quad (13)$$

The value of a_r must be chosen such that it is bounded everywhere during the course of landing; also, it must satisfy the boundary condition of soft landing and hovering. Thus,

$$a_r = u^2[(g_0/u_0^2) - ku + (\xi^2/k_b)(k^2 - \zeta^2)(u - u_0)] \quad (14)$$

where ξ and ζ are positive real constants.

The value of a_r in Eq. (14) is bounded, since the terms u and k are always positive real finite quantities. At the surface of the planet, the value of a_r should be equal to g_0. This is obtainable by substituting $k = 0$ and $u = u_0$ into Eq. (14). The foregoing relation guarantees that the radial acceleration is zero at the point of landing.

3. Solution of the Problem

If Eq. (14) is substituted into Eq. (13), one obtains a linear equation in u:

$$\frac{d^2u}{dk^2} + \left(\frac{1}{k}\right)\frac{du}{dk} + \frac{\beta^2\xi^2}{k^2}(k^2 - \zeta^2)(u - u_0) = 0 \quad (15)$$

or

$$\frac{d^2U}{dk^2} + \frac{1}{k}\frac{dU}{dk} + \left(\omega^2 - \frac{\omega^2\zeta^2}{k^2}\right)U = 0 \quad (16)$$

where

$$\omega = \beta\xi \quad (17)$$

and

$$U = -(u - u_0) = (r - r_0)/rr_0 \quad (18)$$

Under the boundary condition that $U = 0$ at $k = 0$, the solution of Eq. (16) in terms of fractional order Bessel functions of the first kind, as shown in Fig. 2a, is[10–12]

$$\frac{U}{U_b} = \frac{J_{\omega\zeta}[\omega k_b(\theta^2/\theta_b^2)]}{J_{\omega\zeta}(\omega k_b)} \quad (19)$$

where U_b is the value of U at $\theta = \theta_b$.

The derivative of U with respect to k can be shown to be

$$\frac{J_{\omega\zeta}(\omega k_b)}{\omega U_b}\frac{dU}{dk} = \frac{\zeta}{k_b}\left(\frac{\theta}{\theta_b}\right)^{-2} J_{\omega\zeta}\left(\omega k_b\frac{\theta^2}{\theta_b^2}\right) - J_{\omega\zeta+1}\left(\omega k_b\frac{\theta^2}{\theta_b^2}\right) \quad (20)$$

For the Bessel function of $\frac{1}{2}$ order, Eqs. (19) and (20) become simply sine function of θ^2:

$$\frac{U}{U_b} = \left(\frac{\theta}{\theta_b}\right)^{-1}\frac{\sin[\omega k_b(\theta^2/\theta_b^2)]}{\sin(\omega k_b)} \quad (21)$$

and

$$\frac{\sin\omega k_b}{\omega U_b}\left(\frac{dU}{dk}\right) = \left(\frac{\theta}{\theta_b}\right)^{-1}\left[\cos\left(\omega k_b\frac{\theta^2}{\theta_b^2}\right) - \frac{1}{2}\left(\omega k_b\frac{\theta^2}{\theta_b^2}\right)^{-1}\sin\left(\omega k_b\frac{\theta^2}{\theta_b^2}\right)\right] \quad (22)$$

By differentiating Eq. (19) with respect to θ, one obtains

$$\frac{\theta_b J_{\omega\zeta}(\omega k_b)}{2\omega k_b U_b}\frac{dU}{d\theta} = \frac{\theta}{\theta_b}\left[\frac{\zeta}{k_b}\left(\frac{\theta}{\theta_b}\right)^{-2} J_{\omega\zeta}\left(\omega k_b\frac{\theta^2}{\theta_b^2}\right) - J_{\omega\zeta+1}\left(\omega k_b\frac{\theta^2}{\theta_b^2}\right)\right] \quad (23)$$

For $0 < \omega\zeta$, the term $J_{\omega\zeta+1}(0)$ is zero. However, the term $J_{\omega\zeta}$ is of the order of magnitude of $(\theta^2)^{\omega\zeta}$ as $\theta \to 0$. Therefore, the term $dU/d\theta$ is of the order of

$$\theta^{1-2+2\omega\zeta} = \theta^{2\omega\zeta-1} \quad (24)$$

From Eq. (24) and the condition $dr/d\theta = (1/u^2)(dU/d\theta)$, where u^2 is real and positive, it is concluded that, as $\theta \to 0$,

$$dU/d\theta \to \infty \text{ for } 0 < \omega\zeta < \tfrac{1}{2};\ \text{also } dr/d\theta \to \infty, \text{ vertical landing}$$

$$dU/d\theta = \text{const for } \omega\zeta = \tfrac{1}{2};\ \text{also } dr/d\theta = \text{const, inclined landing}$$

$$dU/d\theta = 0 \text{ for } \tfrac{1}{2} < \omega\zeta;\ \text{also } dr/d\theta = 0, \text{ horizontal landing} \quad (25)$$

4. Characteristic Root

The state of the vehicle at the starting point b is defined completely if the quantities k_b, θ_b, U_b, and $(dU/d\theta)_b = W_b$

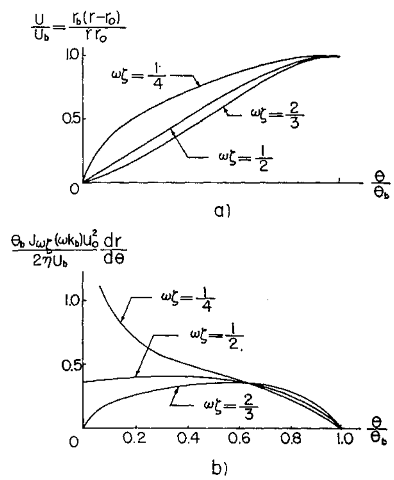

Fig. 2 Position and direction of vehicle as functions of angle-to-go.

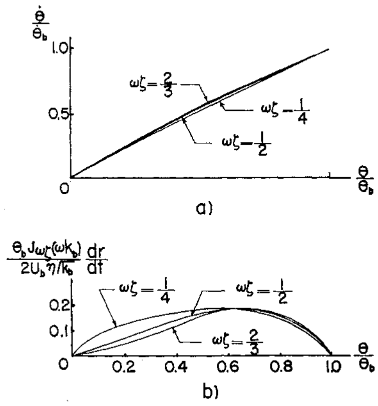

Fig. 3 Angular and radial velocities as functions of angle-to-go.

with respect to the planet are known. From Eq. (23), one has

$$\frac{\theta_b J_{\omega\zeta}(\omega k_b)}{2\omega k_b\, U_b} W_b = \frac{\omega\zeta}{\omega k_b} J_{\omega\zeta}(\omega k_b) - J_{\omega\zeta + 1}(\omega k_b) \quad (26)$$

This equation may be solved for its characteristic root

$$\omega k_b = \eta \quad (27)$$

provided that θ_b, U_b, W_b, and $\omega\zeta$ are given. It is emphasized here that η is the lowest positive real root of Eq. (26).

An example is given here for descent from a circular orbit ($W_b = 0$) along a trajectory of Bessel function of $\frac{1}{2}$ order. Thus, from Eq. (23), one obtains

$$\frac{\theta_b \sin(\omega k_b)}{U_b \omega k_b} \left(\frac{dU}{d\theta}\right)\bigg|_{\theta=\theta_b} = 2\cos\left(\omega k_b \frac{\theta^2}{\theta_b^2}\right) - \left(\omega k_b \frac{\theta^2}{\theta_b^2}\right)^{-1} \sin\left(\omega k_b \frac{\theta^2}{\theta_b^2}\right)\bigg|_{\theta=\theta_b} = 0 \quad (28)$$

from which one has

$$\tan \omega k_b = 2\omega k_b \quad (29)$$

The value $\omega k_b = \eta = 1.16562$ may be obtained for its fundamental root for the case $\omega\zeta = \frac{1}{2}$.

The dimensionless form of $dr/d\theta$ is shown in Fig. 2b for $\omega k_b = \eta$ and various values of $\omega\zeta$.

5. Angular and Radial Velocities

With the aid of Eqs. (4) and (11), the angular velocity $\dot{\theta}$ may be written as

$$\dot{\theta}/\dot{\theta}_b = (\theta/\theta_b)(u^2/u_b^2) \quad (30)$$

From Eqs. (18, 19, and 30), one has

$$\frac{\dot{\theta}}{\dot{\theta}_b} = \left(\frac{u_0}{u_b}\right)^2 \left(\frac{\theta}{\theta_b}\right) \left[1 - \frac{U_b}{u_0} \frac{J_{\omega\zeta}[\omega k_b(\theta^2/\theta_b^2)]}{J_{\omega\zeta}(\omega k_b)}\right]^2 \quad (31)$$

The value of $\dot{\theta}$ at $\theta = 0$ is zero, as shown in Fig. 3a. This is in conformity with the requirement of soft landing. Also,

$$dr/dt = u^{-2}(dU/dt) \quad (32)$$

where

$$\frac{dU}{dt} = \frac{d\theta}{dt}\frac{dU}{d\theta} = k^{1/2}u^2 \frac{dU}{d\theta} \quad (33)$$

Combining Eqs. (11, 23, 32, and 33), one obtains

$$\frac{\theta_b J_{\omega\zeta}(\omega k_b)}{2\omega k_b U_b}\frac{dr}{dt} = \left(\frac{\theta}{\theta_b}\right)^2 \left[\frac{\zeta}{k_b}\left(\frac{\theta}{\theta_b}\right)^{-2} J_{\omega\zeta}\left(\omega k_b \frac{\theta^2}{\theta_b^2}\right) - J_{\omega\zeta+1}\left(\omega k_b \frac{\theta^2}{\theta_b^2}\right)\right] \quad (34)$$

which also is plotted in Fig. 3b for $\omega k_b = \eta$ and various values of $\omega\zeta$.

It is expected that the reasoning given in discussing Eq. (24) holds also here. Thus, the order of magnitude of the term dr/dt is

$$\theta^{2 - 2 + 2\omega\zeta} = \theta^{2\omega\zeta} \quad (35)$$

Therefore,

$$dr/dt|_{\theta=0} = 0 \qquad \text{for } 0 < \omega\zeta \quad (36)$$

Equation (36) indicates that the radial velocity of the vehicle is also zero at the point of contact on the surface of a planet, as would be expected.

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Thermodynamic Calculation of Partly Frozen Flow

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In some analyses of expanding flow with finite chemical kinetics, it is desirable to interpret performance in terms of more than one freezing point. For example, in the expansion of highly dissociated air from a shock tunnel, exact thermokinetic calculations demonstrate that nitric oxide will effectively freeze earlier than atomic oxygen and atomic nitrogen. Thus, in approximate thermodynamic calculations employing fixed assignments of freezing points, it is necessary at some point to be able to provide for nonreactive nitric oxide, i.e., nitric oxide that cannot disappear thermodynamically upon further cooling in compliance with the chemical equilibrium of the reaction $2\ NO = N_2 + O_2$. However, most generalized computer programs involving chemical equilibrium among a mixture of elements necessarily are designed to insure that all possible, individual equilibrium relations are satisfied. If nitric oxide, molecular oxygen, and molecular nitrogen are

Received May 17, 1963.

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